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On Faraday resonance of a viscous liquid

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The tri-diagonal determinant for Faraday resonance of a viscous liquid subject to an externally imposed vertical oscillation is expressed in terms of the surface-wave impedance of the liquid and developed as a continued fraction to obtain a systematic sequence of analytical approximations for the threshold acceleration. The impedance is calculated for either a clean or a fully contaminated surface on the assumption that the capillary, gravitational and viscous length scales are small compared with the breadth and depth of the liquid. Limiting approximations for weak and strong viscosity are constructed.

1. Introduction

I present here a compact formulation of the Faraday-resonance problem (subharmonic free-surface oscillations for a viscous liquid in a container subjected to a vertical oscillation)† in terms of the surface-wave impedance

$$Z(s) = P(s)/N(s), \tag{1.1}$$

where

$$N(s) = \mathscr{L}\eta_k(t) \equiv \int_0^\infty \mathrm{e}^{-st}\eta_k(t)\,\mathrm{d}t \tag{1.2}$$

and $P(s)$ are the Laplace transforms of the free-surface displacement $\eta_k(t)$ and the corresponding, externally imposed specific surface pressure $p_k(t)$ after factoring out the horizontal spatial dependence of wavenumber k. This formulation concentrates the fluid-mechanical part of the problem in the impedance Z (see § 3) and invites comparison with Guillemin's (1957) analysis of ladder networks, which leads to the development of the eigenvalue problem as a continued-fraction sequence independently of the structure of Z. It is significantly more economical than the conventional formulation (e.g. Kumar 1996) that starts from the Navier–Stokes equations.

The externally imposed component of the specific surface pressure for a fluid in a container subject to the acceleration $a = A\cos 2\omega t$ is, from d'Alembert's principle,

$$p_k(t) = -a\eta_k = -A\eta_k\cos 2\omega t. \tag{1.3}$$

The Laplace-transformation of (1.3) is

$$P(s) = -A\mathscr{L}\{\eta_k(t)\cos 2\omega t\} = -\tfrac{1}{2}A[N(s+2\mathrm{i}\omega) + N(s-2\mathrm{i}\omega)], \tag{1.4}$$

† There is an extensive literature (see Miles & Henderson 1990 for a review) on Faraday resonance of a weakly viscous fluid, in which sensible viscous action is confined to thin boundary layers, but explicit results absent this approximation have appeared only recently (Kumar & Tuckerman 1994; Bechhoefer *et al.* 1995; Kumar 1996).

the substitution of which into (1.1) yields

$$Z(s)N(s) + \tfrac{1}{2}A[N(s+2\mathrm{i}\omega) + N(s-2\mathrm{i}\omega)] = 0. \tag{1.5}$$

Subharmonic Faraday resonance occurs for those values of A, k and ω for which η admits a Fourier expansion with the frequencies $\omega_n \equiv n\omega$, $n = 1, 3, \ldots$, and (1.5) admits the corresponding conjugate-imaginary zeros $s = \pm\mathrm{i}n\omega$:

$$Z_n N_n + \tfrac{1}{2}A(N_{n+2} + N_{n-2}) = 0 \quad (n = \pm 1, \pm 3, \ldots), \tag{1.6}$$

where the alternative signs are vertically ordered and the subscript n implies $s = \mathrm{i}n\omega$. The existence of a non-trivial solution of the homogeneous equations (1.6) implies (δ_{mn} is the Kronecker delta)

$$\det\,[\delta_{mn} Z(\mathrm{i}n\omega) + \tfrac{1}{2}A(\delta_{m+2,n} + \delta_{m-2,n})] = 0 \quad (m, n = \pm 1, \pm 3, \ldots) \tag{1.7}$$

for the determination of the threshold acceleration for Faraday resonance.

The implicit hypothesis that the minimum acceleration for instability of the flat surface occurs for subharmonic (rather than harmonic) motion may fail for shallow liquids; see Kumar's (1996) figure 3. However, the appellation 'Faraday' presumably is appropriate only for subharmonic waves.

2. The eigenvalue problem

It is expedient, for the calculation of the zeros of (1.7), to introduce the dimensionless quantities

$$\mathscr{Z}_n \equiv (k/\omega^2)Z(\mathrm{i}n\omega), \quad \varepsilon \equiv \tfrac{1}{2}Ak/\omega^2, \tag{2.1a,b}$$

and transform the tri-diagonal determinant (1.7) to

$$\Delta \equiv \det\,[\delta_{mn}\mathscr{Z}_n + \varepsilon(\delta_{m+2,n} + \delta_{m-2,n})] = 0. \tag{2.2}$$

The eigenvalues of (2.2) may be obtained through standard numerical procedures (Kumar & Tuckerman 1994). Systematic analytical approximations to the dominant eigenvalue may be obtained through the successive truncations

$$\Delta_1 = |\mathscr{Z}_1|^2 - \varepsilon^2, \quad \Delta_3 = |\mathscr{Z}_1\mathscr{Z}_3 - \varepsilon^2|^2 - \varepsilon^2|\mathscr{Z}_3|^2, \tag{2.3a,b}$$

$$\Delta_5 = |\mathscr{Z}_1(\mathscr{Z}_3\mathscr{Z}_5 - \varepsilon^2) - \varepsilon^2\mathscr{Z}_5|^2 - \varepsilon^2|\mathscr{Z}_3\mathscr{Z}_5 - \varepsilon^2|^2, \ldots\ . \tag{2.3c}$$

Dividing these truncations by 1, $|\mathscr{Z}_3|^2$, $|\mathscr{Z}_3\mathscr{Z}_5 - \varepsilon^2|^2, \ldots$, we obtain the continued-fraction sequence (cf. Wall 1948, Ch. XII; Guillemin 1957, §4.2; Chen & Vinals 1997)

$$\varepsilon = |\mathscr{Z}_1|, \quad |\mathscr{Z}_1 - \varepsilon^2\mathscr{Z}_3^{-1}|, \quad |\mathscr{Z}_1 - \varepsilon^2(\mathscr{Z}_3 - \varepsilon^2\mathscr{Z}_5^{-1})^{-1}\ldots|, \tag{2.4a–c}$$

which may be solved by iteration to obtain the sequence

$$\varepsilon = |\mathscr{Z}_1|, \quad |\mathscr{Z}_1(1 - \mathscr{Z}_{13})|, \quad \left|\mathscr{Z}_1\left[1 - \frac{\mathscr{Z}_{13}|1 - \mathscr{Z}_{13}|^2}{1 - \mathscr{Z}_{13}(\mathscr{Z}_1/\mathscr{Z}_5)}\right]\right|, \tag{2.5a–c}$$

wherein $\mathscr{Z}_{13} \equiv \mathscr{Z}_{-1}/\mathscr{Z}_3$. The approximation obtained by solving $\Delta_3 = 0$ as a quadratic in ε^2 is equivalent to Kumar's (1996) equation (3.27); it is algebraically more complicated, but no more accurate, than (2.5b).

3. Surface-wave impedance

Lamb (1932, § 349) considers straight-crested waves of wavenumber k on a deep viscous liquid of kinematic viscosity ν and kinematic surface tension T' and obtains the dispersion function

$$D(s) \equiv kZ(s) = s^2 + 2\alpha s + \sigma^2 + \tau, \tag{3.1}$$

where

$$\alpha \equiv 2\nu k^2, \quad \sigma^2 \equiv gk + T'k^3, \quad \tau \equiv \alpha^2\{1 - [1 + 2(s/\alpha)]^{1/2}\} \qquad (3.2a\text{–}c)$$

($D = kZ$ is more convenient than Z for the description of dispersion, but Z is more directly related to the dynamics). Although derived for two-dimensional motion, (3.1) is isotropic and holds for any $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$ element of a spectral superposition over $\boldsymbol{k}$. The components s^2/k, $2\nu ks$, $g + T'k^2$ and τ/k of the impedance Z represent, respectively, the inertial force associated with the surface wave, the Stokes damping associated with the irrotational component of the flow, the gravitational and capillary restoring forces, and the force (which comprises *negative* damping) associated with the rotational flow.

$D(s)$ admits two zeros in an s-plane cut along $(-\infty, -\nu k^2)$. If $0 < \alpha/\sigma < 2.61$ these zeros are complex conjugates with negative real parts and represent damped capillary–gravity waves. If $\alpha/\sigma > 2.61$, they are negative real with $-s \lesseqgtr 0.70\sigma$ and represent creep/diffusion (cf. Miles 1968, figure 1).

$D(s)$ for finite depth follows from Basset (1888) or Wehausen & Laitone (1960, p. 643). $D(s)$ for a fully contaminated surface, for which the surface condition of vanishing shear stress is replaced by a no-slip condition, is given by (3.1) with (3.2c) replaced by

$$\tau = \tfrac{1}{2}\alpha s\{[1 + 2(s/\alpha)]^{1/2} - 1\}. \tag{3.3}$$

Substituting Z from (3.1) into (2.1a) and introducing the capillary, gravitational and viscous length scales,

$$\ell_c \equiv (T'/g)^{1/2}, \quad \ell_g \equiv g/\omega^2, \quad \ell_\nu \equiv (2\nu/\omega)^{1/2}, \qquad (3.4a\text{–}c)$$

we obtain

$$\mathscr{Z}_n = -n^2 + 2\mathrm{i}\delta n + k\ell_g(1 + k^2\ell_c^2) + \delta^2(1 - r_n^+ - \mathrm{i}r_n^-), \tag{3.5}$$

where

$$\delta \equiv \frac{2\nu k^2}{\omega} = k^2\ell_\nu^2, \quad r_n^\pm(\delta) \equiv \left[\left(\frac{1}{4} + \frac{n^2}{\delta^2}\right)^{1/2} \pm \frac{1}{2}\right]^{1/2}, \qquad (3.6a, b)$$

and $\mathscr{Z}_{-n}$ is the complex conjugate of $\mathscr{Z}_n$. (Note that $\delta n < \operatorname{Im}\mathscr{Z}_n < 2\delta n$.)

Consider, for example, the results presented in figure 6 of Bechhoefer *et al.* (1995) for paraffin oil (which provides a *clean* surface in the present context) at 22 °C, for which $\rho = 0.868$, $g = 981$, $T' = 33.0$ and $\nu = 1.28$, all in c.g.s. units. The present approximations to the threshold acceleration, (2.5a–c), and the corresponding values of $k\ell_c$ are compared in table 1. The maximum truncation error in the first/second approximation is 0.4/0.07%. (The larger errors in $k\ell_c$ are a consequence of $\mathrm{d}A/\mathrm{d}k\ell_c = 0$ at the threshold.)

ω/π (Hz)	$k\ell_c$			a/g		
	(*a*)	(*b*)	(*c*)	(*a*)	(*b*)	(*c*)
20	0.547	0.531	0.531	1.527	1.521	1.521
25	0.716	0.684	0.685	2.391	2.384	2.384
30	0.869	0.818	0.819	3.373	3.369	3.370
35	1.007	0.937	0.938	4.463	4.465	4.466
40	1.133	1.045	1.046	5.653	5.661	5.663
45	1.249	1.142	1.144	6.936	6.952	6.954
50	1.357	1.232	1.234	8.308	8.330	8.334
55	1.457	1.315	1.318	9.765	9.793	9.797
60	1.552	1.393	1.396	11.303	11.335	11.340
65	1.642	1.467	1.470	12.919	12.953	12.960
70	1.728	1.536	1.540	14.609	14.645	14.653
75	1.809	1.602	1.607	16.372	16.406	16.417

TABLE 1. Threshold wavenumber and acceleration for Bechhoefer *et al.*'s (1995) oil of density $0.870\,\mathrm{g\,cm^{-3}}$, surface tension $28.8\,\mathrm{dyne\,cm^{-1}}$, and kinematic viscosity $1.34\,\mathrm{cm^2\,s^{-1}}$. The three columns (a, b, c) for each of $k\ell_c$ and a_1/g are calculated from the successive approximations (2.5*a–c*). The damping parameter δ increases monotonically from 0.36 to 0.87 as ω/π increases from 20 to 75.

4. Limiting approximations

Having established that truncation at $n=1$ typically yields a good approximation to the threshold acceleration, we construct approximations for weak ($\delta \ll 1$) and strong ($\delta \gg 1$) viscosity.

Letting $\alpha/s \to 0$ in (3.1), we obtain (Wehausen & Laitone 1960, §25)

$$D(s) = s^2 + 2\alpha s + \sigma^2 + O(\alpha^{3/2}s^{1/2}) \quad (\alpha/s \to 0). \tag{4.1}$$

The corresponding approximation to (3.5) is

$$\mathscr{L}_n = -n^2 + 2\mathrm{i}\delta n + k\ell_g(1 + k^2\ell_c^2), \tag{4.2}$$

which may be combined with ε (2.1*b*), δ (3.6*a*),

$$\hbar \equiv k\ell_g, \quad \gamma \equiv (\ell_v/\ell_g)^2 = 2\nu\omega^3/g^2, \quad \lambda \equiv (\ell_c/\ell_v)^2 \tag{4.3a–c}$$

in (2.4*a*) to obtain the first approximation

$$A_1/g = 2\hbar^{-1}|\mathscr{L}_1| = 2|1 - \hbar^{-1} + \gamma\lambda\hbar^2 + 2\mathrm{i}\gamma\hbar|. \tag{4.4}$$

The threshold wavenumber and acceleration are determined within $1 + O(\gamma^2)$ by $\mathrm{Re}\,\mathscr{L}_1 = 0$ (cf. resonance of a lightly damped, simple oscillator) and are given by

$$\hbar_s = 1 - \gamma\lambda, \quad A_1/g = 4\gamma(1 - \gamma\lambda), \tag{4.5a,b}$$

which corresponds to the dashed lines in Bechhoefer *et al.*'s figure 6.

The counterparts of (4.1)–(4.3) for $s/\alpha \to 0$ are (Bechhoefer *et al.*)

$$D(s) = \tfrac{3}{2}s^2 + \alpha s + \sigma^2 + O(\alpha^{-1}s^3) \quad (s/\alpha \to 0), \tag{4.6}$$

$$\mathscr{L}_n = -\tfrac{3}{2}n^2 + \mathrm{i}\delta n + k\ell_g(1 + k^2\ell_c^2), \tag{4.7}$$

and

$$A_1/g = 2|1 - \tfrac{3}{2}\hbar^{-1} + \gamma\lambda\hbar^2 + \mathrm{i}\gamma\hbar|. \tag{4.8}$$

The threshold wavenumber and acceleration determined by $\partial A_1/\partial k = 0$ on the hypothesis that $k = O(\gamma^{-1/2})$ admit the asymptotic approximations

$$k_s = (\tfrac{2}{3}\gamma)^{-1/2} + (\tfrac{3}{2}\lambda - 1)\gamma^{-1} + O(\gamma^{-3/2}) \tag{4.9a}$$

and

$$A_1/g = 2(3\gamma)^{1/2} - 2^{3/2}(\tfrac{3}{2}\lambda + 1) + O(\gamma^{-1/2}), \tag{4.9b}$$

but these are useful only for rather large γ (for the largest frequency in table 1, $\omega = 75\pi$, $\gamma = 36.4$ and $\lambda = 2.98$).

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